Persamaan differensial biasa dengan ordo n, merupakan persamaan dengan satu perubah (variabel) yang dapat dituliskan dalam bentuk :

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\ldots, \frac{d^ny}{dx^n}\right) = 0$$

dengan y = f (x)

Penyelesaian persamaan differensial ordo satu dapat lebih dari satu, sehingga untuk mencari penyelesaian yang unik atau khusus memerlukan informasi tambahan berupa **syarat batas**.

Metode untuk penyelesaian Persamaan differensial biasa :

1. EULER'S METHOD

- Deret taylor orde 1
- Sangat sensitif terhadap besarnya "h"

$y_n = y_{n-1} + h \cdot f(x_{n-1}, y_{n-1})$; n = 1,2, 3, ...

$$h = \frac{x_n - x_0}{n}$$

dengan : x_n = nilai x yang ditanya nilai fungsinya.
x_0 = nilai x awal.
n = bilangan bulat

2. MODIFIED EULER'S METHOD

- Mengurangi kesalahan akibat pemilihan "h"

$$y_n^{(k+1)} = y_{n-1} + \frac{h}{2} \cdot \left[f(x_{n-1}, y_{n-1}) + f(x_n, y_n^{(k)})\right]$$

Dengan : $y_n^{(k)} = y_{n-1} + h \cdot f(x_{n-1}, y_{n-1})$
k = 0,1,2,... dan n = 1,2, 3, ...

3. RUNGE-KUTTA METHOD

- Deret taylor orde 4
- Lebih teliti

$$y_{n+1} = y_n + \frac{h}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

dimana : $k_1 = f(x_n, y_n)$
$k_2 = f(x_n + 0{,}5h, y_n + 0{,}5\,h \cdot k_1)$
$k_3 = f(x_n + 0{,}5h, y_n + 0{,}5\,h \cdot k_2)$
$k_4 = f(x_n + h, y_n + h \cdot k_3)$

Contoh :

$\frac{dy}{dx} = y + x$; y(0) = 1,5

Cari nilai y pada x = 0,1 dengan **ketiga metode** di atas
dipilih h = 0,05

1. Euler's Method

Dipilih n = 2

$$h = \frac{x_n - x_0}{n} = \frac{0{,}1 - 0}{2} = 0{,}05$$

Iterasi Pertama (n = 1)

$$\begin{aligned} y_1 = y(0{,}05) &= y_0 + h.\,f(x_0, y_0) \\ &= y_0 + h\,(y_0 + x_0) \\ &= 1{,}5 + (0{,}05)\,(1{,}5 + 0) = 1{,}575 \end{aligned}$$

Iterasi Kedua (n = 2)

$$\begin{aligned} y_2 = y(0{,}1) &= y_1 + h.\,f(x_1, y_1) \\ &= y_1 + h\,(y_1 + x_1) \\ &= 1{,}575 + (0{,}05)\,(1{,}575 + 0{,}05) = 1{,}656250 \end{aligned}$$

Jadi penyelesaian kasus tersebut : **y(0,1) = 1,656250**

2. Modified Euler's Method

- Dengan rumusan Euler's Method

$$\begin{aligned} y_1^{(0)} &= y_0 + h.\,f(x_0, y_0) \\ &= y_0 + h\,(y_0 + x_0) \\ &= 1{,}5 + (0{,}05)\,(1{,}5 + 0) = 1{,}575 \end{aligned}$$

Proses iterasi dilakukan pada rumusan Modified Euler's.
Iterasi Pertama (x_1 = 0,05 dan k = 0) :

$$\begin{aligned} y_1^{(1)} &= y_0 + \frac{h}{2}.\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] \\ &= y_0 + \frac{h}{2}\left[(y_0 + x_0) + (y_1^{(0)} + x_1)\right] \\ &= 1{,}5 + \frac{0{,}05}{2}\left[(1{,}5 + 0) + (1{,}575 + 0{,}05)\right] \\ &= 1{,}578125 \end{aligned}$$

Iterasi Kedua (x_1 = 0,05 dan k = 1) :

$$\begin{aligned} y_1^{(2)} &= y_0 + \frac{h}{2}.\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] \\ &= y_0 + \frac{h}{2}\left[(y_0 + x_0) + (y_1^{(1)} + x_1)\right] \\ &= 1{,}5 + \frac{0{,}05}{2}\left[(1{,}5 + 0) + (1{,}578125 + 0{,}05)\right] \\ &= 1{,}578203 \end{aligned}$$

Iterasi Ketiga (x_1 = 0,05 dan k = 2) :

$$y_1^{(3)} = y_0 + \frac{h}{2}.\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$
$$= y_0 + \frac{h}{2}\left[(y_0 + x_0) + (y_1^{(2)} + x_1)\right]$$
$$= 1,5 + \frac{0,05}{2}\left[(1,5 + 0) + (1,578203 + 0,05)\right]$$
$$= 1,578205$$

Iterasi Keempat (x_1 = 0,05 dan k = 3) :

$$y_1^{(4)} = y_0 + \frac{h}{2}.\left[f(x_0, y_0) + f(x_1, y_1^{(3)})\right]$$
$$= y_0 + \frac{h}{2}\left[(y_0 + x_0) + (y_1^{(3)} + x_1)\right]$$
$$= 1,5 + \frac{0,05}{2}\left[(1,5 + 0) + (1,578205 + 0,05)\right]$$
$$= 1,578205$$

Karena hasil iterasi keempat dan iterasi ketiga (iterasi sebelumnya) sama maka proses iterasi dihentikan dengan dihasilkan harga y_1 = 1,578205

- $$y_2^{(0)} = y_1 + h.\ f(x_1, y_1)$$
$$= y_1 + h\ (y_1 + x_1)$$
$$= 1,578205 + (0,05)\ (1,578205 + 0,05) = 1,659615$$

Iterasi Pertama (x_2 = 0,1 dan k = 0) :

$$y_2^{(1)} = y_1 + \frac{h}{2}.\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$$
$$= 1,578205 + \frac{0,05}{2}\left[(y_1 + x_1) + (y_2^{(0)} + x_2)\right]$$
$$= 1,578205 + \frac{0,05}{2}\left[(1,578205 + 0,05) + (1,659615 + 0,1)\right]$$
$$= 1,662901$$

Iterasi Kedua (x_2 = 0,1 dan k = 1) :

$$y_2^{(2)} = y_1 + \frac{h}{2}.\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$
$$= 1,578205 + \frac{0,05}{2}\left[(y_1 + x_1) + (y_2^{(1)} + x_2)\right]$$
$$= 1,578205 + \frac{0,05}{2}\left[(1,578205 + 0,05) + (1,662901 + 0,1)\right]$$
$$= 1,662983$$

Iterasi Ketiga ($x_2 = 0{,}1$ dan $k = 2$) :

$$y_2^{(3)} = y_1 + \frac{h}{2}.\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right]$$

$$= 1{,}578205 + \frac{0{,}05}{2}\left[(y_1 + x_1) + (y_2^{(2)} + x_2)\right]$$

$$= 1{,}578205 + \frac{0{,}05}{2}\left[(1{,}578205 + 0{,}05) + (1{,}662983 + 0{,}1)\right]$$

$$= 1{,}662985$$

Iterasi Keempat ($x_2 = 0{,}1$ dan $k = 3$) :

$$y_1^{(4)} = y_0 + \frac{h}{2}.\left[f(x_1, y_1) + f(x_2, y_2^{(3)})\right]$$

$$= 1{,}578205 + \frac{0{,}05}{2}\left[(y_1 + x_1) + (y_2^{(3)} + x_2)\right]$$

$$= 1{,}578205 + \frac{0{,}05}{2}\left[(1{,}578205 + 0{,}05) + (1{,}662985 + 0{,}1)\right]$$

$$= 1{,}662985$$

Karena hasil iterasi keempat dan iterasi sebelumnya yaitu iterasi ketiga sama maka proses iterasi dihentikan dengan hasil harga $y_2 = 1{,}662985$

Jadi penyelesaian kasus tersebut : **y(0,1) = 1,662985**

3. Runge-Kutta Method

Dipilih h = 0,05

- Iterasi Pertama ($y_1 = y(0{,}05)$)

$x_0 = 0$; $y_0 = 1{,}5$

$k_1 = f(x_0, y_0) = (y_0 + x_0) = 1{,}5 + 0 = 1{,}5$

$k_2 = f(x_0 + 0{,}5h,\ y_0 + 0{,}5\ h\ .\ k_1)$

$= (y_0 + 0{,}5\ h\ .\ k_1) + (x_0 + 0{,}5h)$

$= (1{,}5 + 0{,}5\ (0{,}05)(1{,}5)) + (0 + 0{,}5\ (0{,}05)) = 1{,}5625$

$k_3 = f(x_0 + 0{,}5h,\ y_0 + 0{,}5\ h\ .\ k_2)$

$= (y_0 + 0{,}5\ h\ .\ k_2) + (x_0 + 0{,}5h)$

$= (1{,}5 + 0{,}5\ (0{,}05)(2{,}275)) + (0 + 0{,}5(0{,}05)) = 1{,}564062$

$k_4 = f(x_0 + h,\ y_0 + h\ .\ k_3)$

$= (y_0 + h\ .\ k_3) + (x_0 + h)$

$= (1{,}5 + (0{,}05)(2{,}6625)) + (0 + 0{,}05) = 1{,}628203$

$$y_1 = y_0 + \frac{h}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$$= 1{,}5 + \frac{0{,}05}{6}(1{,}5 + (2\ .\ 1{,}5625) + (2\ .\ 1{,}564062) + 1{,}628203)$$

$$= 1{,}578178$$

- Iterasi kedua ($y_2 = y(0,1)$)

  $x_1 = 0,05$; $y_1 = 1,578178$

$k_1 = f(x_1, y_1) = (y_1 + x_1) = 1,578178 + 0,05 = 1,628178$

$k_2 = f(x_1 + 0,5h, y_1 + 0,5\,h\,.\,k_1)$

$= (y_1 + 0,5\,h\,.\,k_1) + (x_1 + 0,5h)$

$= (1,578178 + 0,5(0,05)(1,628178)) + (0,05 + 0,5(0,05))$

$= 1,693882$

$k_3 = f(x_1 + 0,5h, y_1 + 0,5\,h\,.\,k_2)$

$= (y_1 + 0,5\,h\,.\,k_2) + (x_1 + 0,5h)$

$= (1,578178 + 0,5(0,05)(1,693882)) + (0,05 + 0,5(0,05))$

$= 1,695525$

$k_4 = f(x_1 + h, y_1 + h\,.\,k_3)$

$= (y_1 + h\,.\,k_3) + (x_1 + h)$

$= (1,578178 + (0,05)1,695525) + (0,05 + 0,05) = 1,776295$

$$y_2 = y_1 + \frac{h}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right] = 1,578178 + \frac{0,05}{6}(1,628178 +$$
$$(2\,.\,1,693882) + (2\,.\,1,695525) + 1,776295) = \mathbf{1,662927}$$